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A RE-ANALYSIS OF MOMENTUM FRACTIONS CARRIED BY QUARKS AND GLUONS IN A MODEL OF PROTON STRUCTURE FUNCTION AT SMALL *x*

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ABSTRACT

This paper reports a re-analysis of a model of proton structure function at small x based on self-similarity. Specifically, it concentrates critically on the pattern of momentum fractions carried by constituent quarks and gluons reported in literature sometime back. Limitation of the approach is also discussed.

Key words: Self-similarity, parton distribution, quarks, gluons

INTRODUCTION

How the quarks and gluons share their longitudinal momentum in proton is an important topic of study by itself. It has been studied in Ref (Gross and Wilczek 1974, Ji et al. 1996, Chen et al. 2009) within perturbative QCD and in Lattice QCD (Deka et al. 2015). It is equally interesting to study the corresponding pattern of such momentum fractions in other phenomenological models of proton available in current literature.

One such model is that of Lastovicka (Lastovicka 2002) based on self-similarity (Mandelbrot 1982) at small *x*. He suggested the self-similarty as a possible feature of multipartons inside a Proton, specifically, in the kinematical region of low Bjorken*x*. Based on this notation, a fundamental form of structure function $F_2(x, Q^2)$ at small *x* was proposed which could explain the H1 (Adloff et al. 2001) and ZEUS (Breitweg et al. 2000) data for $6.2 \times 10^{-7} \leq x \leq 10^{-2}$ and $0.045 \leq Q^2 \leq 120$ GeV^2. While self-similarity is not yet formally established in QCD, it is obtained in renormalization group analysis (Kroger 2000).

Phenomenological validity range of the model of Ref (Lastovicka 2002) is rather limited $6.2 \times 10^{-7} \leq x \leq 10^{-2}$ and $0.045 \leq Q^2 \leq 120$ GeV^2. In Ref (Jahan and Choudhury 2012) such pattern was studied assuming the validity in entire*x*-range $0 < x < 1$ while in Ref (Jahan and Choudhury 2013) it was analyzed for $x_a \leq x \leq x_b$; $x_a = 6.2 \times 10^{-7}$ and $x_b = 10^{-2}$ where momentum fraction carried by quarks $\langle \hat{x} \rangle_q$ and gluons $\langle \hat{x} \rangle_g$ were obtained for partons having momentum fraction between x_a and x_b. The main reason behind the work of Ref (Jahan and Choudhury 2013) is that,it is more reasonable to study the model in the phenomenologically allowed range of *x* than going beyond it as the model(Lastovicka 2002)has a singularity at $x \sim 0.019$, outside the range of validity. One limitation of Ref (Jahan and Choudhury 2012, Jahan and Choudhury 2013) was that the analytical expression for $\langle \hat{x} \rangle_q$ contains an infinite series, but only the leading term was considered without studying their convergence properties. This might make the result unstable and unreliable. The aim of the present communication is to make a re-analysis of Ref (Jahan and Choudhury 2013) and critically examine its stability from the point of view of convergence properties of the infinite series involved. We will show these results are indeed unstable and are sensitive to the number of terms used. We will then use semi-analytical as well as numerical method and obtain stable values. Assuming the validity of the model for $Q^2 \geq 120$ GeV^2,we then obtain scale of Q^2 beyond its phenomenological range of validity where it reaches the QCD asymptotic values (Gross and Wilczek 1974, Ji et al. 1996, Chen et al. 2009)

In section 2,we outline the essential formalism and in section 3, we discuss the results. Section 4 contains the conclusions.

FORMALISM

Momentum Sum Rule

The momentum sum rule is given as (Jahan and Choudhury 2012, Jahan and Choudhury 2013, Close 1979) :

$$\int_0^1 x\sum\{(q_i(x,Q^2)+\bar{q}_i(x,Q^2))\}dx+\int_0^1 G(x,Q^2)dx=1 \qquad (1)$$

Where $G(x,Q^2)dx = xg(x,Q^2)$ (2)

$g(x,Q^2)$is the gluon number density. It can be converted (Jahan and Choudhury 2013) into an inequality if the information about quarks and gluons is available only in a limited range of *x*, say $x_a \le x \le x_b$i.e.

$$\int_{x_a}^{x_b} x\sum\{(q_i(x,Q^2)+\bar{q}_i(x,Q^2))\}dx+\int_{x_a}^{x_b} G(x,Q^2)dx \le 1 \qquad (3)$$

This yields the respective information when the momentum fractions carried by small *x* quarks and gluons in$x_a \le x \le x_b$ to be

$$\langle\hat{x}\rangle_q=\int_{x_a}^{x_b} x\sum\{(q_i(x,Q^2)+\bar{q}_i(x,Q^2))\}dx \qquad (4)$$

$$\langle\hat{x}\rangle_g=\int_{x_a} G(x,Q^2)dx \le 1-\langle\hat{x}\rangle_q \qquad (5)$$

Note that Eqn(5) yields only the upper limit of the fractional momentum carried by the gluons in the regime$x_a \le x \le x_b$. Eqn (1)-(5) shows that the momentum fraction needs information about the parton distribution of the proton within the model.

Self-similar PDF

In Ref (Lastovicka 2002), PDF at small *x* based on self-similarity is defined as in Eqn(3) of Ref (Jahan and Choudhury 2012)

$$q_i(x, Q^2) = e^{D_0^i} f(x, Q^2) \tag{6}$$

Where,

$$f(x,Q^2) = \frac{Q_0^2\left(\frac{1}{x}\right)^{D_2}}{M^2\left\{1+D_3+D_1 log\left(\frac{1}{x}\right)\right\}}\left[\left(\frac{1}{x}\right)^{D_1 log\left(1+\frac{Q^2}{Q_0^2}\right)}\left(1+\frac{Q^2}{Q_0^2}\right)^{D_3+1} - 1\right] \tag{7}$$

is a flavor independent function and D_0^i is the only flavor dependent parameter. M^2(=1 GeV^2) is introduced to make the (PDF) dimensionless. D_1, D_2, D_3 are the model parameters, fitted from HERA data (Adloff et al. 2001, Breitweg et al. 2000):

$$\begin{aligned} &D_1=0.073\pm0.001\\ &D_2=1.013\pm0.01\\ &D_3= -1.287\pm0.01\\ &Q^2= 0.062\pm0.01 \text{ GeV}^2 \end{aligned} \tag{8}$$

In terms of $f(x,Q^2)$, $F_2(x,Q^2) = e^{D_0} x f(x,Q^2)$ (9)

Where,

$$e^{D_0} = \sum_{i=1}^{n_f} e_i^2\left(e^{D_0^i} + e^{\bar{D}_0^i}\right) \tag{10}$$

With D_0 = 0.339 $\pm$ 0.145

In terms of structure function the momentum sum rule inequality is

$$\int_{x_a}^{x_b}\{a\,F_2(x,Q^2)+G(x,Q^2)\}dx\ \leq 1 \tag{11}$$

Where $a=\frac{e^{\tilde{D}_0}}{e^{D_0}}$ is Q^2-independent parameter determined from data (Sloan, Smadja and Voss 1988), a =3.1418, using the fractionally charged quarks.

Analytical Expression of $\langle\hat{x}\rangle_q$ and its limitations

The analytical expression of is given as[Eqn(23) of Ref (Jahan and Choudhury 2013)]

$$\langle\hat{x}\rangle_q= \quad \frac{e^{\widetilde{D}_0}Q_0^2}{D_1M^2}e^{\left(\frac{1+D_3}{D_1}\right)(2-D_2)}\left\{\left(1+\frac{Q^2}{Q_0^2}\right)^{D_3+1}e^{-\left(\frac{1+D_3}{D_1}\right)D_1log\left(1+\frac{Q^2}{Q_0^2}\right)}I_1-I_2\right\} \tag{12}$$

Where the integrals I_1 and I_2 are expressible in terms of infinite series

$$I_i=\int\frac{e^{\mu_i z}}{z}dz=\log|z|+\sum_{i=1}^{\infty}\frac{\mu_i z^n}{n.n!}\qquad ;\ i=1,\ 2 \tag{13}$$

$$z=\frac{1+D_3}{D_1}+log\frac{1}{x} \tag{14}$$

And

$$\mu_1=\ D_1\log\left(1+\frac{Q^2}{Q_0^2}\right)+D_2-1 \tag{15}$$

$$\mu_2=D_2-1 \tag{16}$$

In Ref (Jahan and Choudhury 2013) only the 1st term of the infinite series is taken into account without taking into account the convergence property and their Q^2-dependence. Below, we address to this point.

RESULT AND DISCUSSION

Q^2-dependence of the convergence of the infinite series

The integral I_1 is Q^2-dependent while I_2 is not, as can be seen from Eqn(15) and (16) above. Convergent condition between nth and (n-1)th term of the infinite series is

$$\frac{\mu_i^{(n-1)}.z^{(n-1)}}{(n-1).(n-1)} \gg \frac{\mu_i^n.z^n}{n.n!} \quad ; \quad i = 1,2 \tag{17}$$

Leading to

$$z \ll \frac{n^2}{(n-1)}.\frac{1}{\mu} \tag{18}$$

It can be explicitly seen that if one includes more and more terms in the infinite series I_1, the convergent condition shifts to higher values of Q^2. As an illustration, the relative convergence taking respectively the ratios of the 3rd vs 2nd term, 4th vs 3rd term, 5th vs 4th, 6th vs 5th term results in the inequalities as

$$\log\left(1+\frac{Q^2}{Q_0^2}\right) \ll 15.384 \tag{19a}$$

$$\ll 45.454 \tag{19b}$$

$$\ll 52.631 \tag{19c}$$

$$\ll 58.823 \tag{19d}$$

These inequalities saturates at 2.9×10^5, 3.4×10^{18}, 4.4×10^{21}, 2.1×10^{24}GeV^2 respectively which are far above the experimentally accessible

HERA range 3×10^4GeV^2 (Aaron et al. 2010). However, it is the slow convergence of the two infinite series which makes the result highly unstable.

Table 1: Values of $\langle \hat{x} \rangle_q$ with higher order terms in $I_1$and I_2 for different Q^2

Q^2(GeV^2)	$\langle \hat{x} \rangle_q$(n=1)	$\langle \hat{x} \rangle_q$(n=2)	$\langle \hat{x} \rangle_q$(n=3)	$\langle \hat{x} \rangle_q$ (n=4)	$\langle \hat{x} \rangle_q$(n=5)
$Q^2 = Q_0^2$	3.700x10^{-2}	-1.630x10^{-1}	5.070x10^{-1}	-1.164	2.170
10	2.781x10^{-1}	-9.520x10^{-1}	2.527	-4.950	8.107
20	3.163x10^{-1}	-1.037	2.694	-5.169	8.360
40	3.582x10^{-1}	-1.112	2.830	-5.329	8.533
60	3.750x10^{-1}	-1.150	2.897	-5.399	8.605
80	3.911x10^{-1}	-1.176	2.939	-5.455	8.660
100	4.037x10^{-1}	-1.194	2.969	-5.467	8.672

In column 2 of Table 1, we record the result of Reference (Jahan and Choudhury 2013), taking only one term of the infinite series. In the same table, we now show the corresponding results taking 2, 3, 4, 5 terms of the two infinite series. In column 3, all $\langle \hat{x} \rangle_q$are negative. From column 4, it is seen saturation occurs below Q^2=10 GeV^2 and in column 5, again all $\langle \hat{x} \rangle_q$ are negative while in column 6, saturation occurs even below $Q^2 = Q_0^2$GeV^2.

Semi-analytical and Numerical Result

As a consequence of the limitation of the analytical method, we take recourse to semi analytical method i.e. we evaluate I_1 and I_2 numerically. Here I_2 is -independent and is obtained as .

Table 2: Values of $\langle\hat{x}\rangle_q$ with I_1 and $\langle\hat{x}\rangle_g$ for different Q^2 using semi-analytical method

Q^2(GeV^2)	I_1	$\langle\hat{x}\rangle_q$	$\langle\hat{x}\rangle_g$
$Q^2 = Q_0^2$	1.076×10^{-2}	1.941×10^{-4}	9.998×10^{-1}
10	6.026×10^{-2}	4.020×10^{-3}	9.959×10^{-1}
20	7.975×10^{-2}	5.527×10^{-3}	9.944×10^{-1}
40	1.059×10^{-1}	7.549×10^{-3}	9.924×10^{-1}
60	1.250×10^{-1}	9.026×10^{-3}	9.909×10^{-1}
80	1.407×10^{-1}	1.023×10^{-2}	9.897×10^{-1}
120	1.669×10^{-1}	1.226×10^{-2}	9.877×10^{-1}
5.62×10^4	3.0558	2.355×10^{-1}	7.645×10^{-1}
3.61×10^5	9.223	7.123×10^{-1}	2.877×10^{-1}
6×10^5	12.870	~1	~0

In Table 2, we record the numerical values of I_1 together the values of $\langle\hat{x}\rangle_q$. From above, it is seen I_1 and I_2, the two infinite series are positive definite and hence $\langle\hat{x}\rangle_q$ and $\langle\hat{x}\rangle_g$ also. Column 3 represents the $\langle\hat{x}\rangle_q$ and column 4 represents the upper limit of $\langle\hat{x}\rangle_g$. $\langle\hat{x}\rangle_q$ saturates at $Q^2 = 6 \times 10^5$ GeV^2. It is interesting to compare the corresponding saturation scale $Q^2 = 5.43 \times 10^6$ GeV^2 of the 1st term of the infinite series of Ref (Jahan and Choudhury 2013) which is of course found to be unstable.

Table 3: Numerical values of $\langle\hat{x}\rangle_q$ and $\langle\hat{x}\rangle_g$ for different Q^2

Q^2(GeV^2)	$\langle\hat{x}\rangle_q$	$\langle\hat{x}\rangle_g$
$Q^2 = Q_0^2$	2.114×10^{-3}	9.978×10^{-1}
10	2.158×10^{-2}	9.784×10^{-1}
20	2.595×10^{-2}	9.740×10^{-1}
40	3.094×10^{-2}	9.690×10^{-1}
60	3.426×10^{-2}	9.657×10^{-1}
80	3.662×10^{-2}	9.633×10^{-1}
100	3.867×10^{-2}	9.613×10^{-1}
120	4.038×10^{-2}	9.596×10^{-1}
580	5.831×10^{-2}	9.416×10^{-1}
3.9×10^6	~ 1	~ 0

In Table 3, we record the numerical values $\langle\hat{x}\rangle_q$ and $\langle\hat{x}\rangle_g$ of model of Ref (Lastovicka 2002) for a few representative values of Q^2using Eqn (20) and (21)instead of Eqn (12) .

$$\langle\hat{x}\rangle_q = \int_{x_a}^{x_b} aF_2(x,Q^2)dx \tag{20}$$

$$\langle \hat{x} \rangle_g = 1 - \langle \hat{x} \rangle_q \qquad (21)$$

From Tables 2 and 3, we observe that in the present model, as Q^2 increases $\langle \hat{x} \rangle_q$ too increases while $\langle \hat{x} \rangle_g$ decreases. However, unlike QCD, the model cannot be extrapolated beyond $Q^2 = 3.9 \times 10^6$ GeV^2, its saturation limit.

Comparison with standard QCD asymptotic

We note that in Ref (Gross and Wilczek 1974, Ji et al. 1996) the asymptotic QCD predictions of $\langle x \rangle_q$ and $\langle x \rangle_g$ are:

$$\lim_{Q^2 \to \infty} \langle x \rangle_q = \frac{3n_f}{2n_g + 3n_f} \qquad (22)$$

$$\lim_{Q^2 \to \infty} \langle x \rangle_g = \frac{2n_g}{2n_g + 3n_f} \qquad (23)$$

Here, n_f and n_g represent the number of active flavors and number of gluons respectively. For SU(3)$_c$, $n_g = 8$. In Ref (Chen et al. 2009), it has alternative asymptotic prediction:

$$\lim_{Q^2 \to \infty} \langle x \rangle_q = \frac{6n_f}{n_g + 6n_f} \qquad (24)$$

$$\lim_{Q^2 \to \infty} \langle x \rangle_g = \frac{n_g}{n_g + 6n_f} \qquad (25)$$

While Eqn (22) and (23) implies that except for $n_f = 6, \langle x \rangle_q < \langle x \rangle_g$. Eqn (24) and (25) indicates the opposite asymptotic feature $\langle x \rangle_q > \langle x \rangle_g$. In the above Eqn

(22-25), $\langle x \rangle_q$ and $\langle x \rangle_g$ denotes the momentum fractions carried by quarks and gluons respectively for *x*-range $0 < x < 1$.

However, still it will be interesting to calculate the momentum scale Q^2 at which the model prediction of partial momentum fractions carried by quarks $\langle \hat{x} \rangle_q$ coincides with the corresponding asymptotically predicted momentum fractions $\langle x \rangle_q$ in standard QCD. This is shown in Table 4.

Table 4: Values of momentum scale Q^2 of the model corresponds to flavored dependent asymptotic QCD predictions of $\langle x \rangle_q$

n_f	$\langle x \rangle_q$	Q^2(GeV^2)
3 Ref [1,2]	9/25	3.8×10^5
4 Ref [1,2]	3/7	5.5×10^5
5 Ref [1,2]	15/31	7.5×10^5
6 Ref [1,2]	9/17	9.3×10^6
3 Ref [3]	9/13	1.76×10^6
4 Ref [3]	3/4	2.13×10^6
5 Ref [3]	15/19	2.37×10^6
6 Ref [3]	9/11	2.56×10^6

Table 4 shows the results of asymptotic values of $\langle x \rangle_q$. As an illustration, asymptotic value of $\langle x \rangle_q$ = 15/19 of Ref (Chen et al. 2009) is achieved at $Q^2 = 2.37 \times 10^6$ GeV^2.

CONCLUSION

In this paper, we have reported a re-analysis of momentum fractions carried by quarks and gluons in a model of proton structure function based on self-similarity (Lastovicka 2002). It is an improvement of an earlier analysis done by us (Jahan and Choudhury 2012, Jahan and Choudhury 2013). In standard QCD (Gross and Wilczek 1974)), momentum fraction carried by quarks decreases, while in the present model, it increases. However, even if such prediction appears to be unwarranted, it is closer to the alternative asymptotic QCD behavior predicted in Ref (Chen et al. 2009)$\langle x\rangle_q > \langle x\rangle_g$.

Let us conclude the paper with a few comments.

The notion of self-similarity, although very interesting, is not yet established in perturbative QCD: the experimental study during last decade has not yet confirmed this idea. Of course, some constraints from general approach such a unitarity, analyticity and in particular the Froissart theorem (Froissart 1961) can be suitably incorporated in a self-similar proton as been done in Reference (Jahan and Choudhury 2014). Our more recent study (Saikia and Choudhury 2014) further indicates that a singularity free version of the self-similarity might be true only at low Q^2,non-perturbative region. Besides, in the present analysis, the analysis has been extrapolated far beyond its phenomenological range of validity to compare with QCD asymptotic while it is not justified rigorously. We have reported such analysis in the present work for completeness. Currently, we are further pursuing the model to make it singularity free, compatible with logarithmic rise of structure function with Q^2. At high Q^2, Froissart Saturation at very small *x* is suitably extrapolating to large *x* limit so that $F_2(x,t) \to 0$ as (1-*x*))$\to 0$.

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